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Uniformity Theorems in Missile Duels

Joel Spencer

A Report prepared for

UNITED STATES AIR FORCE PROJECT RAND

Rand
SANTA MONICA, CA 90406

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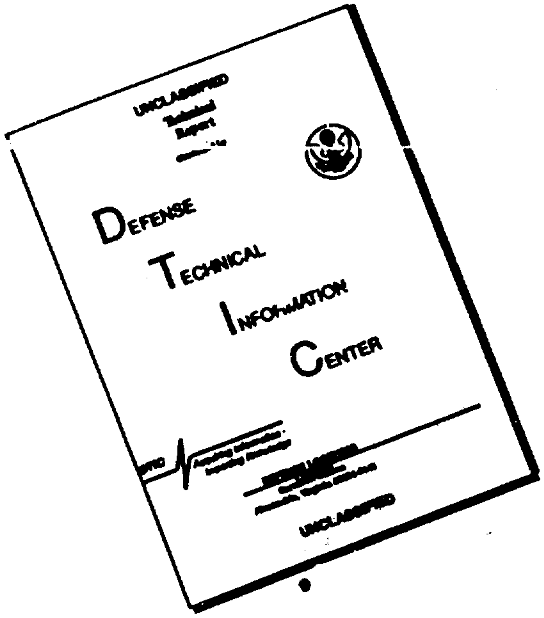
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10. ABSTRACT

A theorem dealing with the purchase of defensive weapon systems is proven. The purchaser's objective is assumed to be the achievement of an assumed destruction criterion at minimal cost. It is shown that under certain circumstances it is optimal to purchase a uniform defensive system. The applicability of the mathematical theorem to the real world is then studied. A second theorem gives a simple procedure for finding an attacker's optimal firing rule under very general circumstances. An example illustrates the problems in choosing a defensive weapon system.

11. KEY WORDS

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PREFACE

The Rand program of strategic studies involves the study of new concepts and doctrine for the strategic forces. As part of this study the effect of various methods of preserving the land-based missile force on long-term force posture stability is being considered. This report establishes certain mathematical theorems which will be of assistance in calculating the stability properties of active defense of missile silos. It should be of interest primarily to weapon system analysts.

SUMMARY

In this report, a theorem dealing with the purchase of defensive weapon systems is proven. We assume the purchaser's objective is to achieve an "assured destruction" criterion at minimal cost. We show that under certain circumstances it is optimal to purchase a uniform defensive system. The applicability of the mathematical theorem to the "real world" is then studied.

A second theorem gives a simple procedure for finding an attacker's optimal firing rule under very general circumstances.

An example is given which illustrates the problems in choosing a defensive weapons system.

CONTENTS

1. A THEOREM ON UNIFORM MISSILE BASING

Our model consists of two sides - Red and Blue. Both sides place missiles in targets (which we may think of as silos) which they may defend in various ways. Each side must meet an "assured destruction" (AD) requirement. That is, their position must be such that after an all-out attack by the other side they expect to have a certain predetermined number of missiles surviving. (This number is usually defined as sufficiently large to inflict "unacceptable" losses on the enemy's cities. However, the calculation of this number is of no concern to us in this note.) We shall assume that Blue's posture is fixed. We assume that Red's object is to meet its AD requirement at minimal cost. We shall prove that under certain quite general conditions Red should have uniform basing. That is, Red could, without penalty, build a set of identically defended targets.

Let us suppose Red may purchase N different kinds of targets. For $1 \leq i \leq N$ set c_i = the cost of the ith type target. (We may think intuitively of the ith type being a missile and (i-1) interceptors.) Red's posture is then given by an ordered N-tuple $P = (n_1, \ldots, n_N)$ which represents the purchase of n_i targets of type i, $i = 1, \ldots, N$. The cost to Red is $c(P) = \Sigma\, c_i n_i$.

Blue may attack a target by sending a nonnegative number of warheads at the target. For $1 \leq i \leq N$ set p_{ij} = the probability that a target of type i is destroyed when attacked by j warheads. So $p_{i0} = 0$, and $0 \leq p_{ij} \leq 1$. We make no other conditions on the p_{ij}.

An attack A by Blue is given by nonnegative A_{ij}, $1 \leq i \leq N$, j nonnegative. Here A_{ij} represents the number of targets of type i that have j missiles sent against them. Thus an attack A on a position P must satisfy

$$(1.1) \qquad \sum_j A_{ij} = n_i \text{ for } 1 \leq i \leq N$$

Set $\mathcal{A}(P)$ = the set of possible attacks A on P (i.e., the set of A satisfying (1)). Set

$$w(A) = \sum_{i,j} j\, A_{ij}$$

= the number of warheads used in attack A. Set

$$(1.2) \qquad \mathcal{A}_W(P) = \{A \in \mathcal{A}(P) : w(A) = W\}.$$

In reality, the n_i and A_{ij} must be integral. However, in this model, we allow them to range over the nonnegative

reals. We return to this point in Section 2.

A Blue attack $A \in \mathcal{A}(P)$ against a Red position P will leave an expected number

$$s(A,P) = \sum_{i,j} A_{ij}(1-p_{ij})$$

of Red targets surviving. Let Blue have a fixed number, W, of warheads. Let $D \geq 0$ be fixed. A position P of Red is said to meet an assured destruction requirement D if for all $A \in \mathcal{A}_W(P)$, $s(A,P) \geq D$. Set $T = T(D,W)$ = the set of positions that meet this requirement. $T \neq \emptyset$ as, for example, $(W + D, 0, \ldots, 0) \in T$. Red wishes to find the position $P \in T$ of minimal cost.

We first show that such a P exists.* If $P = (n_1, \ldots, n_N) \in T$ has some $n_i > W + D$ then replacing that n_i by $W + D$ gives a $P \in T$ of less cost. Therefore, we may restrict P to $T' = T \cap \{(n_1, \ldots, n_N) : n_i \leq W + D \text{ for all } i\}$.

*This paragraph shows that it is impossible that $P \in R$ exist with cost "infinitesimally" greater than some C_0 without some $P \in R$ existing with cost C_0. The reader unfamiliar with the topological concepts involved may assume the results as obvious and continue.

As s is a continuous function, T is a closed set. Therefore T' is a compact set. The cost function is continuous and therefore assumes its minimum on the compact set T'.

Set $L = L(D,W)$ = the set of positions of least cost. L has at least one member. We call a position P uniform if only one of the n_i is nonzero. That is, P is uniform if all targets are of the same type.

<u>Theorem</u>: There exists a uniform P in L.

<u>Proof</u>: We first note that if λ_i are nonnegative reals and $A_i \in \mathcal{A}(P_i)$ then $\Sigma \lambda_i A_i \in \mathcal{A}(\Sigma \lambda_i P_i)$ and $s(\Sigma \lambda_i A_i, \Sigma \lambda_i P_i) = \Sigma \lambda_i s(A_i, P_i)$. Fix a $P = (n_1, \ldots, n_N) \in L$. Set $C = c(P) = \Sigma c_i n_i$ and set $S = \{i: n_i \neq 0\}$. For each $i \in S$ let P_i be the position with $n_i = C/c_i$ and all other $n_j = 0$. Then $c(P_i) = C$ and P_i is uniform. If the theorem is not true the P_i must all fail to meet the AD requirements. There would exist attacks $A_i \in \mathcal{A}(P_i)$ such that $s(A_i, P_i) < D$. Setting $\lambda_i = c_i n_i / C$ and $A = \Sigma \lambda_i A_i$ we have $P = \Sigma \lambda_i P_i$, $A \in A(P)$ and $s(A,P) = \Sigma \lambda_i s(A_i, P_i) < D \, \Sigma \lambda_i < D$ as $\Sigma \lambda_i = \Sigma c_i n_i / C = 1$. But then $P \notin L$, a contradiction.
Q.E.D.

2. ASSUMPTIONS AND RESERVATIONS

In this section we delineate the assumptions upon which Theorem 1 was proven. We also note some situations in which the theorem does not apply.

(1) The theorem assumes there are no area defense possibilities. By area defense we mean defensive stratagems that defend more than one of the targets simultaneously--for example, as might occur with "long range" interceptors.

(2) We have assumed that the targets contain identical missiles. If, on the other hand, some targets contain more powerful missiles than others it is likely that the more powerful missiles should receive more protection.

(3) We have allowed n_i and A_{ij} to assume nonintegral values. The theorem is false if we require n_i and A_{ij} to be integral, as they must be in reality. Here is a counter-example: Blue has $W = 3$ warheads and Red has AD requirement $D = 1$. There are $N = 2$ options for Red. $p_{10} = 0$, $p_{11} = 1$, $p_{20} = p_{21} = p_{22} = 0$, $p_{23} = 1$. Intuitively, Blue must use 1 warhead to knock out a target of type 1 and 3 warheads to knock out a target of type 2. $c_1 = 1$, $c_2 = 2$. The best uniform positions that meet the AD requirement are (4, 0) and (0, 2) both of cost 4. However, (1, 1) meets the AD requirement with cost 3. Against (1, 1) Blue would like

to knock out the type 1 target with 1 warhead and 2/3 type 2 targets with 2 warheads. He is prevented by the integrality of the targets.

We feel this is not an important flaw in the theorem. We have in mind applications with hundreds (or more) of missiles. If the optimal strategy calls for, say, 156.4 missiles it can easily be approximated by 156 or 157 missiles with no serious loss.

It would be quite a more serious matter to allow the number of warheads aimed at one target to be nonintegral. An optimal strategy of 2.5 warheads/target might not easily be approximated by a mix of 2 and 3 warheads/target. It would be especially dangerous to approximate a strategy of .5 warheads/target by a mix of 0 and 1 warhead/target. Note that we have assumed that the number of warheads/target is integral.

(4) The theorem assumes that Red starts from scratch. If, for example, Red starts with a number of missiles in unprotected silos the theorem does not apply. It might then be best for Red to protect some of the missiles with interceptors and leave the rest unprotected. If Red was able to instantaneously scrap what position he had and sell it for full cost the theorem would hold. In reality,

practically the complete opposite is true. When a weapon system is discarded there is only a negligible return. Full deposit, no return is closer to the truth.

(5) The previous point leads to an important model. Let D be fixed and let Blue gradually increase his number of warheads, W. For any fixed value of W Red should have a uniform defense of type, say t(W). As the example in Section 4 illustrates t(W) is not a constant function. Thus Red might find he has all defenses of type 1 and suddenly type 2 is optimal. This gives an example of the necessity of predicting the future value of your opponent's W.

(6) In the model we assume that the values of the p_{ij} are known. In reality it is difficult, if not impossible, to obtain reliable estimates of the p_{ij}. In reality, weapons systems are put into operation for long periods of time. Even if reasonable estimates of the p_{ij} are available for the current time the effect of future research can only be guessed at. This suggests a model based on Bayesian distributions.

In a Bayesian model we have, instead of a fixed value of p_{ij}, a probability distribution for the p_{ij}. We give an extreme, though instructive, example. Suppose a type 1 defense is totally impregnable against the enemy's current

warheads. However, you suspect the enemy is working on countermeasures which, if successful, would enable him to destroy your target with a single missile. You give him a 50% chance of finding these countermeasures in the appropriate time period. In our original model what would p_{11} be? Setting $p_{11} = 0$ or 1 is clearly wrong. Also, setting $p_{11} = .5$ (the average) does not reflect the realities. We instead give p_{11} a distribution saying Prob $(p_{11} = 0) =$ Prob $(p_{11} = 1) = .5$.

In this model the expected value usage in the assured destruction requirement would have to change. A reasonable requirement would be that Red would have to have a 90% chance of meeting its AD requirement. For example, let $D = 100$ and Blue have $W = 1000$ warheads. Suppose Red has three options. Each option is currently impregnable. For each option there is a 40% chance that some new technology will leave them completely vulnerable--i.e., each target could be knocked out with a single warhead. Assume these 40% probabilities are independent--that is, the target defenses are so different that research that would enable one kind of defense to be destroyed will not help destroy other defenses. Say all defenses have unit cost. Then a uniform position would cost Red 100. For if Red had

< 1100 targets there would be a 40% chance that 1000 of his targets could be knocked out, leaving him with less than his AD requirement. However, Red can meet his requirement at cost 600 by buying 200 of each weapon. Red will meet his AD requirement unless Blue can counter all three of his defenses, which he can do with probability $.4 \cdot .4 \cdot .4 = .064$, so Red meets his AD requirement with probability 93.6%.

3. OPTIMAL ATTACK

In this section we assume that Red has n targets, all identically defended. Blue has W warheads. We set p_i = the Kill Probability if i warheads are sent at a target. We want to know the optimal attack for Blue--that is, the attack which maximizes the expected number of Red losses.

It will help to consider an example. Let p_i be given by

i	0	1	2	3	4	5	6
p_i	0	.1	.5	.6	.9	.95	1

We place the points (i, p_i) on a graph and draw the convex hull of the points. (See Fig. 3.1.) Given p_0, p_1, ... we define the Attack Sequence as the sequence $0 = a_0 < a_1 < a_2 < \ldots$ of integers such that (a_i, p_{a_i}) is in the convex hull of $\{(n, p_n)\}$. In the example, 0, 2, 4, 5, 6 is the attack sequence. We have "cut off" p_i at $i = 6$ in the example although, for theoretical purposes, it may be easier to consider p_i defined for all integers i.

Proposition: If j is not in the attack sequence then Blue should never (for any W) attack any targets with j warheads.

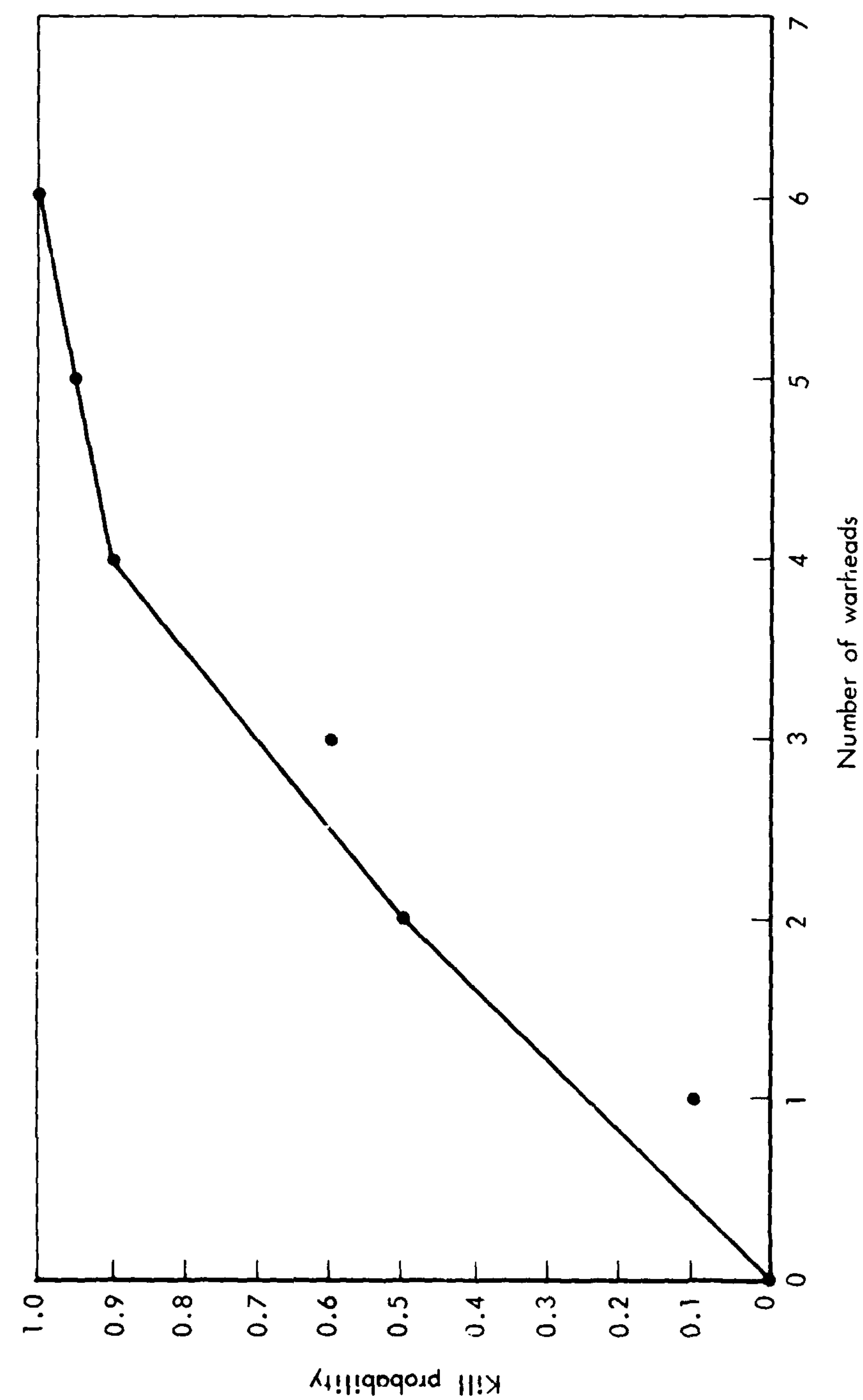

Fig.3.1 — Attack sequence example

Proof: As (j, p_j) is not on the convex hull there exist i, k,

$$(3.1) \qquad i < j < k, \quad p_j < p_i[\frac{k - j}{k - i}] + p_k[\frac{j - i}{k - i}].$$

Now suppose Blue attacks x targets at force level j. Blue could instead attack $x[\frac{k - j}{k - i}]$ targets with i warheads and $x\ [\frac{j - i}{k - i}]$ targets with k warheads. This replacement is valid as we still have x targets being attacked by jx warheads. But now the expected number of targets destroyed $= x\ [p_i(\frac{k - j}{k - i}) + p_k(\frac{j - i}{k - i})] > xp_j$, the previous number. Thus any attack using force levels not in the attack sequence is suboptimal.

Proposition: If $i < j < k$ are in the attack sequence and (i, p_i), (j, p_j), (k, p_k) are not collinear then Blue should never attack simultaneously at force levels i and k.

Proof: We have

$$(3.2) \qquad p_j > p_i[\frac{k - j}{k - i}] + p_k[\frac{j - i}{k - i}]$$

the strict inequality due to the noncollinearity. Say Blue attacks $x_i > 0$ targets at force level i and $x_k > 0$ targets

at force level j. Set $y = \min[x_i(k - i)/(k - j)$, $x_k(j - i)/(k - i)]$. Blue can replace the attack of $y[\frac{k - j}{k - i}]$ targets with i weapons and $y\ [\frac{k - i}{j - i}]$ targets with k weapons with an attack of y targets with j weapons. This new attack will, by (3.2), have a greater expected number of targets destroyed and thus the original attack was suboptimal.

There is a remaining possibility that $i < j < k$ are in the attack sequence with (i, p_i), (j, p_j), (i, p_k) collinear. In this case there may be a continuum of optimal attacks as an attack at level j may be replaced by attacks at level i, k with equal effect. If, for some W, an optimal attack uses force level j it may be replaced by an optimal attack that does not. We define the reduced attack sequence by deleting j if there exist i, k in the attack sequence $i < j < k$, (i, p_i), (j, p_j), (k, p_n) collinear. That is, i is in the reduced attack sequence iff (i, p_i) is a corner of the convex hull of $\{n, p_n\}$. In the example 0, 2, 4, 6 is the reduced attack sequence.

<u>Optimal Attack Algorithm</u>: Let the p_i be fixed and let $0 = a_0 < a_1 < \ldots$ be the reduced attack sequence. We consider Blue's optimal attack as W increases from 0.

At $W = 0$ all targets are attacked at level 0. As W increases, targets are attacked at level a_1 until all have been attacked ($W = na_1$). Then the force level is increased to a_2 on more and more targets until $W = na_2$ and all have been so attacked. In general, for $na_i \leq W \leq na_{i+1}$ all targets are attacked by at least a_i warheads and as many as possible $[(W - na_i)/(a_{i+1} - a_i)]$ are attacked by a_{i+1} warheads.

We note that, using the optimal attack algorithm, if the Red targets stay constant and W increases Blue never needs to retarget his warheads.

We shall use the following two results in Section 4.

If $p(i) = p_i$ is (strictly) convex then the (reduced) attack sequence is 0, 1, 2, 3, ... In particular, if $p_i = 1 - \alpha^i$, $0 < \alpha < 1$ (which is the case with i warheads each with independent kill probability $1 - \alpha$) the attack sequence is 0, 1, 2, 3, ...

If $p(i) = p_i$ is (strictly) convex for $i \geq i_0$ and $j \geq i_0$ is in the (reduced) attack sequence then j, $j + 1$, $j + 2$, ... are in the attack sequence.

4. AN ILLUSTRATIVE EXAMPLE

We assume that Red has N = 3 different kinds of targets he may purchase. The Kill Probabilities and Costs are given by:

$$
\begin{array}{llll}
p_{1i} = 1 - .5^{i} & & & c_1 = 1 \\
p_{2i} = 1 - .75^{i} & & & c_2 = 1.5 \\
p_{3i} = 1 - .9^{i}, & i = 0, 1 & & c_3 = 1.2 \\
\quad\; 1 - .9^{2}.2^{i-2} & i \geq 2 & &
\end{array}
$$

(See Table 4.1) We have thought of the first type target as being a moderately strong silo, so that the Blue warheads have KP = .5. The second type target is a hard rock silo so that Blue KP = .25. The third type target is equipped with two interceptors. The Blue KP = .1 when the Blue warhead is attacked by an interceptor, but = .8 when the Red interceptors are depleted. Let us suppose Red must meet an "Assured Destruction" requirement of D = 100 targets. We shall find the Red position of minimal cost that meets this AD requirement for each number W of Blue warheads.

Table 4.1

KILL PROBABILITIES AGAINST TARGETS

Missiles	Target 1	Target 2	Target 3
0	.0000	.0000	.0000
1	.5000	.2500	.1000
2	.7500	.4375	.1900
3	.8750	.5781	.8380
4	.9375	.6836	.9676
5	.9688	.7627	.9935
6	.9844	.8220	.9987
7	.9922	.8665	.9997
8	.9961	.8999	.9999
9	.9980	.9249	1.0000
10	.9990	.9437	1.0000
11	.9995	.9578	1.0000
12	.9998	.9683	1.0000
13	.9999	.9762	1.0000
14	.9999	.9822	1.0000
15	1.0000	.9866	1.0000
16	1.0000	.9900	1.0000
17	1.0000	.9925	1.0000
18	1.0000	.9944	1.0000
19	1.0000	.9958	1.0000

We know from convexity properties that against targets 1 and 2 the attack sequence is 0, 1, 2, 3, 4, ... whereas against target 3 the attack sequence is 0, 3, 4, 5, ... For each target type we graph in Fig. 4.1 the fraction of targets destroyed versus W/n. To do this, note that if $W/n = a_i$ is in the attack sequence then the fraction of targets destroyed is p_i. T`e graph is then gotten by linear interpolation.

Now for each target type i and each warhead number W we ask for the minimal number of targets n so that D will survive. We set $q_i(W/n)$ = the fraction of targets not destroyed when W warheads attack n targets. So q_i is a piecewise linear function and we need solve

$$nq_i(W/n) = D.$$

The left-hand side is monotonically increasing in n so we find attack levels a_{k-1}, a_k so that

$$a_{k-1}q_i(W/a_{k-1}) < D \leq a_k q_i(W/a_k).$$

Then we know

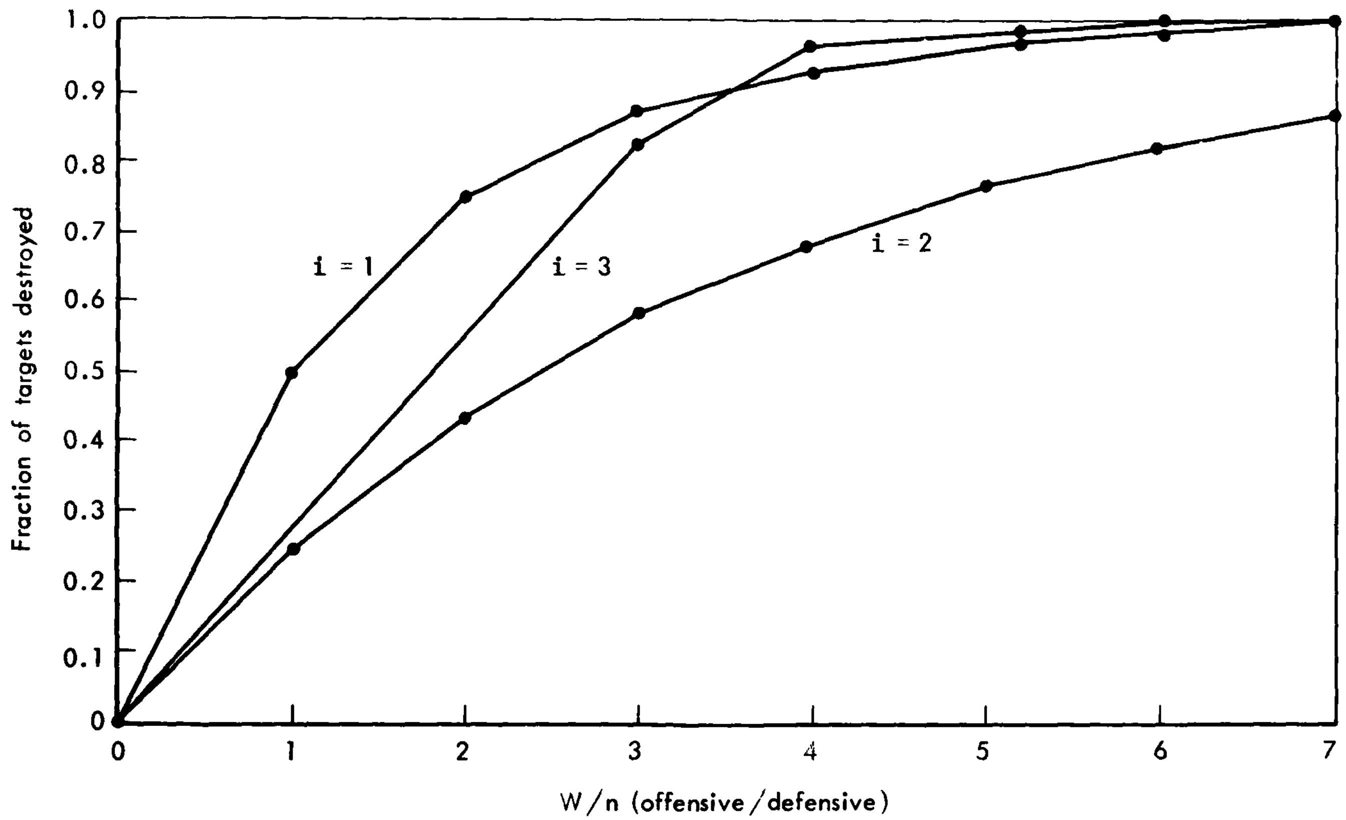

Fig.4.1 — Targets destroyed in optimal uniform attack

$$W/a_{k-1} \leq n \leq W/a_k .$$

In this region, q_i is linear so we can solve for

$$n = \frac{(a_k - a_{k-1})D - (q_k - q_{k-1})W}{q_{k-1}a_k - q_k a_{k-1}} .$$

The cost is given by nc_i. These calculations are very simply done on the computer giving the results of Table 4.2.

Note how the optimal target type does change as a function of W. For $0 \leq W \leq 120$, target defense 1 is optimal. From W = 120 to W = 680 (approximately) target defense 3 is optimal. From W = 680 to W = 1000 (and beyond) target defense 2 is optimal.

An interesting anomaly occurs if we ignore defense 2. Then defense 3 is optimal for $120 < W < 920$ whereas defense 1 is optimal for $W < 120$ and $W > 920$. That is, as W increases, defense 1 loses its optimality and then later regains it.

There are a number of factors that must come into the choice of target defense:

(1) The current and expected number of enemy warheads. For as we have seen, the optimal defense depends on the number of enemy warheads.

Table 4.2

UNIFORM DEFENSE COST AGAINST W WARHEADS

W	Target 1	Target 2	Target 3
20	110.0000	157.5000	126.7040
40	120.0000	165.0000	133.4080
60	130.0000	172.5000	140.1120
80	140.0000	180.0000	146.8160
100	150.0000	187.5000	153.5200
120	160.0000	195.0000	160.2240
140	170.0000	202.0000	166.9280
160	180.0000	208.0000	173.6320
180	190.0000	214.0000	180.3360
200	200.0000	220.0000	187.0400
220	206.6667	226.0000	193.7440
240	213.3333	232.0000	200.4480
260	220.0000	238.0000	207.1520
280	226.6667	244.0000	213.8560
300	233.3333	250.0000	220.5600
320	240.0000	256.0000	227.2640
340	246.6667	262.0000	233.9680
360	253.3333	267.7778	240.6720
380	260.0000	272.7778	247.3760
400	266.6667	277.7778	254.0800
420	273.3333	282.7778	260.7840
440	280.0000	287.7778	267.4880
460	286.6667	292.7778	274.1920
480	293.3333	297.7778	280.8960
500	300.0000	302.7778	287.6000
520	306.6667	307.7778	294.3040
540	313.3333	312.7778	301.0080
560	320.0000	317.7778	307.7120
580	326.6667	322.7778	314.4160
600	333.3333	327.7778	321.1200
620	340.0000	332.7778	327.8240
640	346.6667	337.7778	334.5280
660	353.3333	342.7778	341.2320
680	360.0000	347.7778	347.9360
700	366.6667	352.7778	354.6400
750	383.3333	363.8889	371.4000
800	400.0000	374.6032	388.1600
850	412.5000	385.3175	404.9200
900	425.0000	396.0317	421.6800
950	437.5000	406.7460	438.4400
1000	450.0000	417.4603	455.2000
1050	462.5000	428.1746	471.9600
1100	475.0000	438.8889	488.7200
1150	487.5000	449.6032	505.4800
1200	500.0000	460.3175	522.2400

(2) The refundability and convertibility of different defenses. As we have noted, the enemy warheads may increase to a point where the current defense type is suboptimal. It is usually impossible to get a sizeable fraction of the target cost back when discarding it. It might be best to keep the suboptimal defenses in place and build new defenses of a different type. This greatly complicates the analysis as now we must consider attacks on nonuniform targets. If, for example, target i is a missile with i - 1 interceptors then it may be possible to convert target i to target j ($i < j$) for the same total cost as buying target j initially. This situation could be considered equivalent to total refundability.

(3) The effect of your missiles on enemy plans. Thus far we have considered targets as purely defensive. However, in reality they contain warheads. Building targets will induce the enemy to build more targets himself with more warheads and thus set off an arms race. It may prove better to use a suboptimal defense which uses fewer targets so as to dampen the arms race. The analogous situation for Deceptive Basing Postures was considered in [1].

The author hopes to discuss points (2) and (3) more thoroughly in a future work.

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1. Brown, T. A. and Joel Spencer, Long-Term Stability of Deceptive Basing Postures, R-650-PR, December 1970. (Unclassified--For Official Use Only).